Hamburg Lectures On Spectral Networks — Lecture 1

September 10, 11 & 13

Lecture 1: Basic Brane Background

1. Motivation
2. Dp-branes & YMH
3. Geometrization of Higgs mechanism
4. D4 in T^*C: Hitchin systems
5. M-theory
6. M5 in T^*C: Class S
7. Coulomb branch of $d=4$, $\mathcal{N}=2$ F.T.
8. Recovering Seiberg-Witten theory
9. Relation to Hitchin Moduli Space

1. Motivation

As the token physicist at this school I thought the best thing I could bring to the table is some description of the physical background and intuition that led a small community of string theorists to the study of Higgs bundles and Hitchin systems in their attempt to understand a collection of 4-dimensional Quantum Field Theories known as "Class S theories."

I will mostly be describing a point of view developed with:

Davide Gaiotto & Andy Neitzke

in a series of six papers we wrote between 2008 and 2012. Of course, there are many other groups with related but different viewpoints, including work done here in Hamburg by Jörg Teschner and his group.

Some Suggested Sources

String theory, M-theory and Branes

1. Joe Polchinski, String Theory, vols. 1+2
2. C. Johnson, D-Brane Primer hep-th/0007170
3. C. Johnson, D-Branes, Cambridge, 2003

G. Moore "What is a brane?" - Notices AMS

G. Moore "D-branes 101" - ITP Lectures 1998

Review of GMN: Moore: 2012 Felix Klein Notes & Videos of 10 lectures: Talk 47 on homepage

Four-Dim'l N=2 Field Theory & Physical Mathematics 1211.2331

Neitzke short reviews: Hitchin Systems ... 1412.7120

Cluster-like coordinates .... PNAS

N=2 d=4 Field Theories: Y. Tachikawa,

N=2 Supersymmetric Dynamics For Pedestrians 1312.2684

These notes:

www.physics.rutgers.edu/~gmoore/ talk #84

More specifically, the goal will be to explain some of the physical background to the construction of Spectral networks

Let C be a Riemann surface and consider a branched cover:

$$\pi : \Sigma \longrightarrow C \qquad \Sigma \subset T^*C$$

We can give a local foliation of C by choosing a phase $e^{i\vartheta}$ and a pair of branches and writing

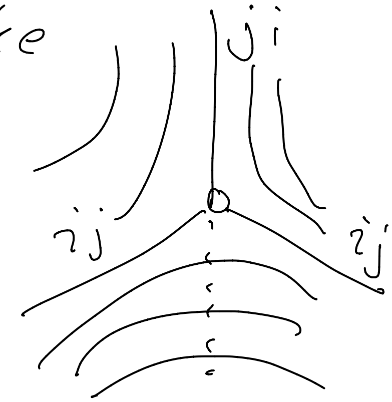
$$\langle \partial_t, \lambda_i - \lambda_j \rangle = e^{i\vartheta}$$

∂_t = tangent vector along paths in the foliation

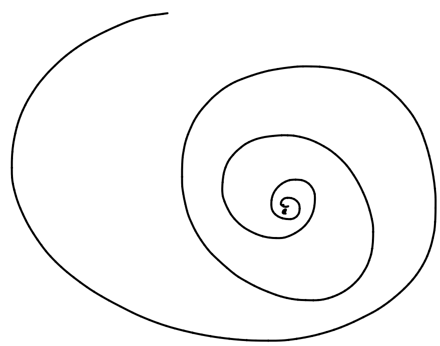
λ_i, λ_j = Restriction of Liouville form λ on T^*C to branches i, j

Near a simple branch point

The foliation looks like



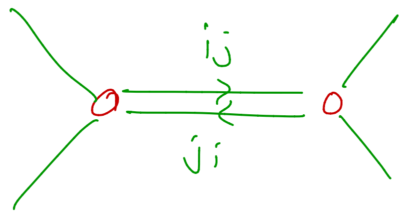
At punctures λ has a singularity and the punctures serve as a basin of attraction.

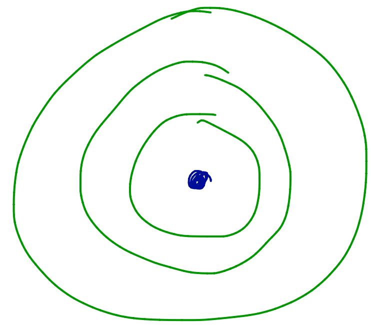


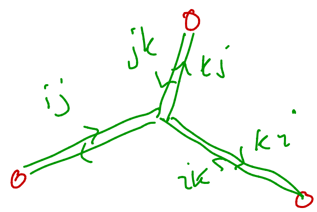
By following the 3 trajectories out of each branch point and applying some local rules when paths collide we generate a graph.

This graph is the spectral network W_ϑ

At special angles ϑ_* this graph contains <u>finite WKB curves</u> :







These lift to closed curves on Σ and correspond to BPS states of the 4D class S theory associated to C.

Counting such states we get "BPS invariants"

BPS invts are $\mathbb{Z}$-valued functions on (charge) lattices - these lattices are often K-theory lattices. They are closely related to DT invariants

So spectral networks actually give an algorithm for computing these BPS invts.

Moreover, they allow us to define local holomorphic symplectic coordinates on Hitchin moduli space.

A natural set of holomorphic functions is defined by

$$\mathrm{Tr}_R\left(\mathrm{Pexp}\oint_{\wp}\mathcal{A}\right)$$

$\mathcal{A}$ - flat $G_{\mathbb{C}}$ connection on $E \longrightarrow C$

$\wp$ — closed path in C

R - f.d. rep$^{\underline{n}}$ of $G_{\mathbb{C}}$

Viewing Hitchin moduli space in complex structure $\zeta \in \mathbb{C}^*$ as a character variety:

$\exists$ Expansion ("Darboux expansion")

$$\mathrm{Tr}_R\left(\mathrm{Pexp}\oint_{\wp}\mathcal{A}\right) = \sum_{\Gamma} \overline{\underline{\Omega}}(R,\wp,\gamma,\zeta)\, \mathcal{Y}_\gamma$$

Here $\overline{\underline{\Omega}}(R, \wp, \gamma)$ is another kind of integer-valued BPS counting function related to "framed BPS states."

$\mathcal{Y}_\gamma$ = monomials in the SN coordinates

$$\mathcal{Y}_{\gamma_1} \mathcal{Y}_{\gamma_2} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \mathcal{Y}_{\gamma_1 + \gamma_2}$$

Relating to Hitchin system

$$\mathcal{A} = \zeta^{-1} \varphi + A + \zeta \overline{\varphi}$$

$$F(A) + [\varphi, \overline{\varphi}] = 0 \quad \text{etc.}$$

$\mathcal{Y}_\gamma$ well-suited to $\zeta \longrightarrow 0$ asymptotics and indeed there is a close relation between SN's and Stokes' rays.

2. Dp Branes & YMH

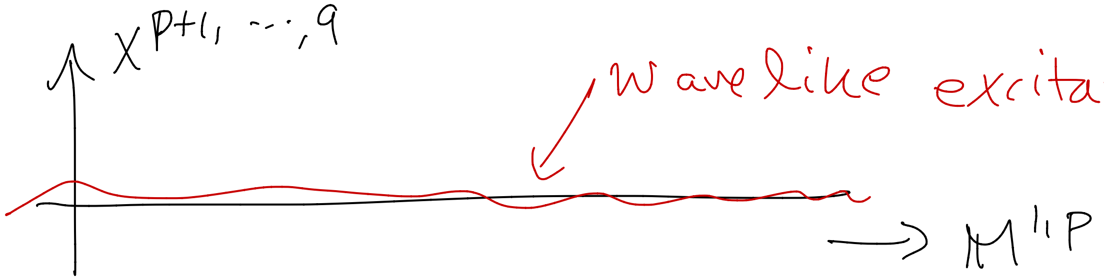
Type II string theory is a physical theory of strings moving in 10-dim'l spacetime.

At low energies it is described by a 10-dim'l supergravity field theory (+ ∞ series of corrections by massive fields)

The equations of motion admit solitonic objects called branes, which have their own dynamical "internal" degrees of freedom.

In their supersymmetric ground state in 10-dim'l Minkowski space p-branes span hyperplanes, WLOG $(X^0, X^1, \cdots X^9) \in M^{1,9}$

$$X^{p+1} = X_0 \quad \cdots\cdots \quad X^9 = X_0$$



Draw vertically for consistency later

The wavelike excitations are described by a (p+1)-dim'l field theory - just like we might describe the height of water waves in the ocean by a 2+1 dim'l field theory of a real scalar field (the height of the water waves).

For Dp-branes, the field theory is the "dim'l reduction of 10D U(N) SYM"

$N=1$: "Single brane"

$N>1$: "Stack of branes"
(for reasons explained below)

What is 10D SYM?

Choose a compact Lie group G.

10D G-SYM is a field theory defined on smooth, oriented, spin, Lorentzian 10-manifolds $\mathcal{X}_{10}$

To define the fields we introduce a principal G- bundle

$$P \longrightarrow \mathcal{X}_{10}$$

Then the bosonic fields are just a connection on P so locally

$$\nabla = d + A \qquad A = A_M dx^M \qquad A_M \in \mathfrak{g}$$

The fermionic fields ("gauginos") are

$$\lambda \in \Gamma(S^+ \otimes \mathrm{ad}P)$$

S^+ = real rank 16 chiral spin bundle

As a quantum (Low Energy Effective) theory if $\mathcal{X}_{10} = \mathbb{R} \times \mathcal{X}_9$ then we have a $\mathbb{Z}_2$-graded Hilbert space

$$\mathcal{H}(\mathcal{X}_9) = \mathcal{H}^0(\mathcal{X}_9) \oplus \mathcal{H}^1(\mathcal{X}_9)$$

and a collection of susy operators

$Q(\epsilon)$ odd Hermitian operators

depending linearly on

$\epsilon \in \text{Cov. Const}_{\nabla^{LC}} \subset \Gamma(S^+)$

part of SUSY algebra $\left(Q(\epsilon)\right)^2 \sim H$

Field multiplet: $\{Q(\epsilon), \lambda\} = \underbrace{\Gamma^{MN}\epsilon\, F_{MN}}$

Clifford mult. of curvature by ϵ.

<u>Def</u>: a.) A supersymmetric ground state is a solution of $Q(\epsilon)|\Psi\rangle = 0$ for some ϵ. "Number of preserved susy's" is the $\mathbb{R}$-dimension of space of such ϵ's.

b.) A supersymmetric classical field configuration is a gauge field s.t.

$\{Q(\epsilon), \lambda\} = 0$ for some ϵ.

Relation between these notions:

$$0 = \langle \Psi | \{ Q(\epsilon), \lambda \} | \Psi \rangle$$

$$= \langle \Psi | \Gamma^{MN}_{\epsilon} F_{MN} | \Psi \rangle$$

Here F_{MN} is an operator in QFT.
But there is a notion of a "coherent state"
Useful in semiclassical analysis where

$$\langle \Psi | F_{MN} | \Psi \rangle \approx F_{MN}^{\text{classical}}$$

Now the equation:

$$\exists \epsilon \qquad \Gamma^{MN}_{\epsilon} F_{MN}^{\text{class}} = 0$$

is an interesting PDE on ∇:

For example: In 4d; if ϵ is chiral

$$\gamma^{\mu\nu}_{\epsilon} F_{\mu\nu} = 0 \iff F + *F = 0$$

ASD equations

Returning to our D_p - brane located at

$$X^{p+1, \dots, 9} = 0$$

The Low energy field theory is the <u>dimensional</u> <u>reduction</u> of 10D U(N) SYM restricted to the worldvolume $W_{p+1} \cong M^{1,p}$

"Dimensional reduction" means:

Truncate to field configurations translation invariant in normal directions $\frac{\partial}{\partial x^{p+1}}, \dots, \frac{\partial}{\partial x^9}$

$$\underbrace{A_M}_{M=0,\dots,9} \longrightarrow \underbrace{A_\mu}_{\mu=0,\dots,p} \qquad \underbrace{A_a}_{a=p+1,\dots,9}$$

A_a: $N \times N$ Hermitian Scalar fields on W_{p+1}

And they are only functions of $x^{0,\dots,p}$;

$$A_\mu(x^0, \dots, x^p), \quad A_a(x^0, \dots, x^p)$$

In more general spacetimes

$$\mathcal{W}_{p+1} \overset{i}{\hookrightarrow} \mathcal{X}_{10}$$

There are supersymmetry conditions on the embedding i and low energy excitations are described by a SUSY field theory consisting of

1. G-connection on $P \longrightarrow \mathcal{W}_{p+1}$
2. Section of normal bundle $\mathcal{N}$

$$X \in \Gamma(\mathcal{W}_{p+1}, \mathcal{N} \otimes \mathrm{ad}\, P)$$

3. $\lambda \in \Gamma\left((S(T\mathcal{W}) \otimes S(\mathcal{N}))^{+} \otimes \mathrm{ad}\, P \right)$
4. Susy param's: $\epsilon \in \Gamma[(S(T\mathcal{W}) \otimes S(\mathcal{N}))^{+}]$

<u>Note:</u> Spin(9-p) symmetry of rotations in normal directions becomes structure group of normal bundle and acts as an "R-symmetry" of the (p+1)-dim'l SUSY gauge theory.

Note that X_a, λ involve sections of the normal bundle in their def$^{\underline{n}}$.

To write down kinetic terms or supersymmetric variations we must choose a connection ∇^R on $\mathcal{N}$. It is called an "R-symmetry connection" because the $Spin(9-p)$ structure group of $\mathcal{N}$ plays the role of an "R-symmetry" in the $(p+1)$-dim'l worldvolume theory.

(An "R-symmetry" is just a global symmetry that does not commute with the supercharges.)

If there is to be unbroken susy, i.e. if cov. const. spinors ϵ are to exist then we must have a reduction of structure group of $T\mathcal{W}_{p+1}$ and $\mathcal{N}$ and an isomorphism between sub-bundles so that under this isomorphism

We have

$$\underbrace{\nabla^{LC}}_{\text{Conn. on } TW_{p+1}} \cong \nabla^{R}$$

This is a (partial) topological twisting condition. In general it only guarantees that some components of $T_{\mu\nu}$ are Q-exact.

In particular note that the normal bundle "scalars"

$$X \in \Gamma(\mathcal{N} \otimes \mathrm{ad}P)$$

will pick up some tensorial properties on W_{p+1}: They are no longer scalars.

3. Geometrization of the Higgs Mechanism

Let us return to $\mathcal{X}_{10} = \mathbb{M}^{1,9}$ and $W_{p+1} \cong \mathbb{M}^{1,p}$ embedded linearly.

Let us also consider $N=1$, a single brane.

Then A_a enter the Lagrangian with no potential energy: both classically and quantum-mechanically they can take a vev:

$$\langle A_a \rangle$$

Now, properly speaking

$$X_a = l_s^2 A_a$$

is the section of the normal bundle and l_s is the string length.

The interpretation of

$$\langle X_a \rangle = x_a \qquad a = p+1, \ldots, 9$$

is clear: It is the position of the brane in the normal space

$$M^{1,P} = W_{p+1} \oplus W$$

Indeed $S = \int d^{p+1}x \; \cdots + (\partial X_a)^2 + \cdots$

and X_a is the Goldstone boson for broken translation invariance.

What about $N > 1$

Now $u(N) \cong su(N) \oplus u(1)$

and the $u(1)$ measures the Center of mass

$$\langle X_a \rangle = \begin{pmatrix} x_a & & \\ & \ddots & \\ & & x_a \end{pmatrix}$$

again x_a measures the position of the brane with $N > 1$.

But now we can notice something curious:

The bosonic part of the action is (I am not careful about some factors of 2):

10D SYM
$$\frac{1}{g_s l_s^6}\int_{M^{1,9}} d^{10}x \, \mathrm{Tr}\, F_{M_1M_2}F^{M_1M_2}$$

$\Downarrow$ Dimensional Reduction

$$\frac{1}{g_s l_s^{p-3}}\int_{W_{p+1}} d^{p+1}x \, \mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}+l_s^{-4}(D_\mu X_a)^2+l_s^{-8}[X_a,X_b]^2\right\}$$

$$D_\mu X_a = \partial_\mu X_a + [A_\mu, X_a]$$

If the X_a are simultaneously diagonalizable and constant along W_{p+1} we still have vanishing energy!

So, in the SYM on $\mathcal{W}_{p+1}$ we can have VEV's which, in a suitable gauge are: $A_\mu = 0$ and:

$$\langle X_a \rangle = \begin{pmatrix} X_a^{(1)} & & \\ & \ddots & \\ & & X_a^{(N)} \end{pmatrix}$$

with $X_a^{(i)}$ not necessarily $= X_a^{(j)}$

and these are classically zero energy.

Using the supersymmetry we can show this family of exact groundstates persists in the quantum theory.

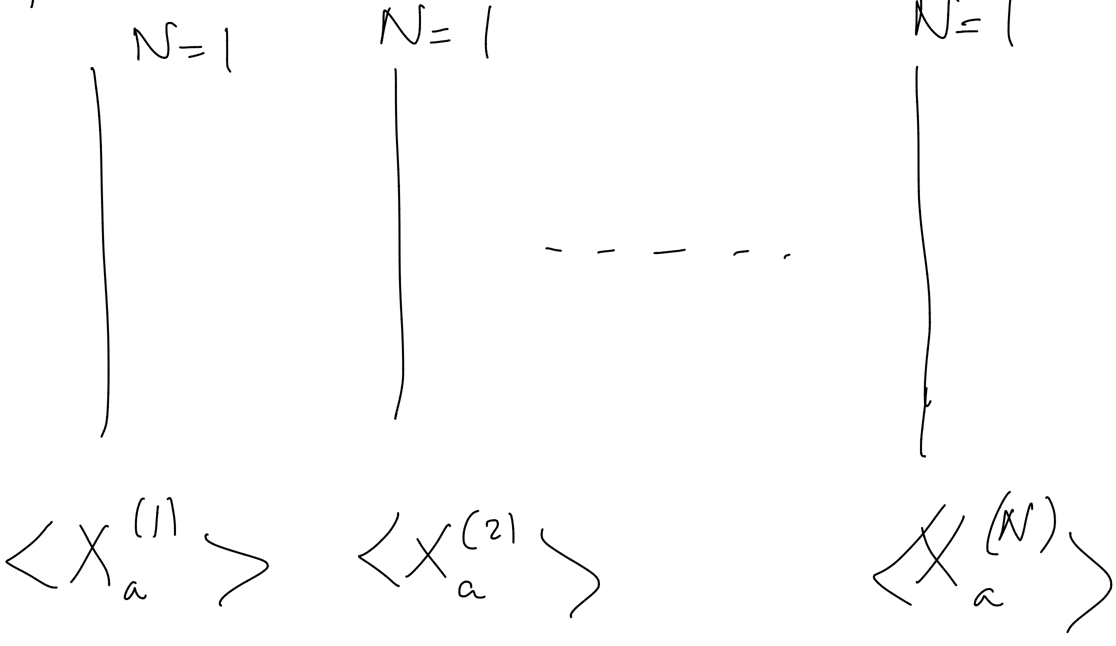
Note that since X_a transforms by conjugation under $U(N)$ gauge transformations for *generic* $X_a^{(i)}$ the symmetry is spontaneously broken

$$u(N) \longrightarrow u(1) \oplus \cdots \oplus u(1)$$

$$U(N) \longrightarrow U(1)^N$$

There is a very striking D-brane interpretation of these vacua (Witten, hep-th/9510135)

We note that parallel branes in $M^{1,9}$ preserve the same amount of susy:



The same cov. const. ϵ works for all.

We interpret the susy vacuum of the $U(N)$ theory on $\mathcal{W}_{p+1}$ with

$$\langle \bar{X}_a \rangle = \begin{pmatrix} x_a^{(1)} & & \\ & \ddots & \\ & & x_a^{(N)} \end{pmatrix}$$

as the configuration of parallel branes at the indicated positions.

That's why we refer to $N>1$ as a "Stack of D-branes."

<u>Remark:</u> Note that the normalizer of T, $N(T)$ preserves the diagonal condition and

$$1 \to T \longrightarrow N(T) \longrightarrow S_N \to 1$$

This S_N can be interpreted as permuting the branes.

2.) The positions in space in the transverse directions are only well-defined in the vacuum. The X_a are really quantum fields. This is highly suggested of some kind of "noncommutative geometry."

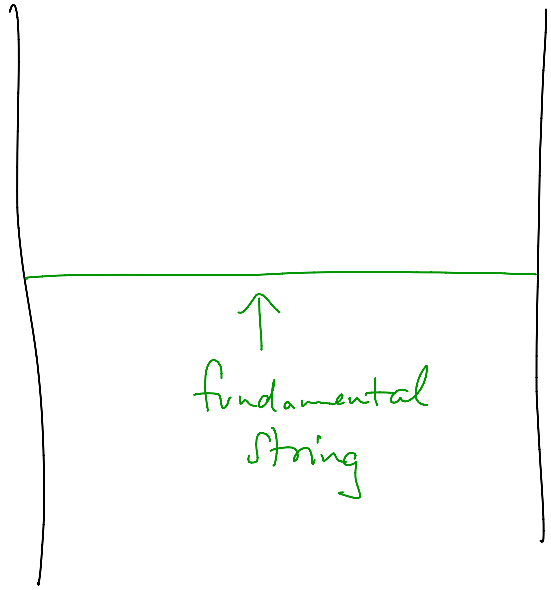
Exercise: Decompose the off-diagonal gauge fields as

$$A_\mu = \sum_{i \neq j} A_\mu^{ij} e_{ij}$$

and show that A_μ^{ij} has a mass M^{ij}:

$$(M^{ij})^2 = l_s^{-4} \sum_a \left(X_a^{(i)} - X_a^{(j)} \right)^2$$

Remark: This has a beautiful brane interpretation:



$$\langle X_a^{(i)} \rangle \qquad\qquad \langle X_a^{(j)} \rangle$$

- The fundamental string carries tension l_s^{-2}. So a string stretched between the i^{th} and j^{th} branes has energy:

$$l_s^{-2} \sqrt{\sum_a \left(\vec{X}_a^{(i)} - \vec{X}_a^{(j)} \right)^2}$$

- There are, in fact, no quantum corrections to this formula because it is an example of a "BPS state."

- The fundamental string is also a brane. It is a 1-brane. Note that branes can end on branes.

- The moduli space of supersymmetric vacua

$$\mathcal{M}_{Coulomb} = \mathrm{Sym}\left(\mathbb{R}^{9-p} \otimes \mathfrak{t}\right)$$

$\mathfrak{t}$ = Cartan subalgebra of $\mathfrak{u}(N)$

is called the "Coulomb branch" because, at a generic point, the gauge symmetry is spontaneously broken to an <u>abelian</u> gauge symmetry and the force law for Abelian Maxwell theory between two stationary electric charges is known as "Coulomb's law."

In class S, the analogous Coulomb branch of vacua will be the base of the Hitchin fibration.

4. D4 In T^*C: Hitchin Systems

Now consider IIA in 10D spacetime:

$$\mathcal{X}^{10} = \mathbb{M}^{1,2} \times T^*C \times \mathbb{R}^3_{7,8,9}$$

C = Riemann surface with metric (possibly with punctures)

product metric: Mink $\oplus$ HK $\oplus$ Eucl. $\Rightarrow$ Some susy preserved.

N D4's located @
- zero section of T^*C
- $\vec{X}_{7,8,9} = 0$

So $\mathcal{W}_5 = \mathbb{M}^{1,2} \times C$

The normal bundle splits naturally

$$\mathcal{N}(\mathcal{W}_5 \hookrightarrow \mathcal{X}_{10}) = \mathcal{N}_2 \oplus \mathcal{N}_3$$

$$\mathcal{N}_2 = \mathcal{N}(C \hookrightarrow T^*C) \qquad \mathcal{N}_3 \cong \mathbb{R}^3_{7,8,9}$$

So we indeed have a reduction of structure group:

$$\mathrm{Spin}\,5 \longrightarrow \mathrm{Spin}\,2 \times \mathrm{Spin}\,3$$

Moreover on $\mathcal{X}_{10}$ there is an 8-dim'l space of covariantly constant spinors since T^*C, regarded as a HK manifold has two covariantly constant spinors for a rank 4 spin bundle — that is, we preserve half the supersymmetries. The other factors are flat so we preserve 1/2 of the original 16 supersymmetries for 8 supersymmetries.

In this case the HK structure on T^*C gives us the desired identification

$$\underset{\text{on } W_2}{\nabla^{R}} \cong \underset{\text{on } TC}{\nabla^{LC}}$$

leading to a partial topological twisting.

We expect the theory to be

- Independent of Kähler structure of C
- Dependent on complex structure on C

There is again a moduli space of vacua of the 5D SYM on

$$W_5 = M^{1,2} \times C$$

and the supersymmetry equations are the Hitchin equations for $U(N)$ on C:

$$F + [\phi, \phi^{+}] = 0$$

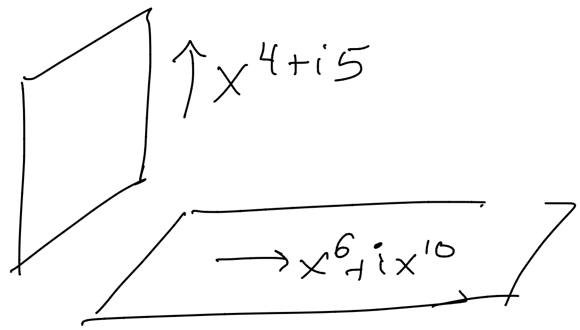
$$\bar{\partial}_A \phi = 0$$

ϕ is part of the normal bundle scalars.

We are choosing coordinates

$$\underbrace{\mathbb{M}^{1,2}}_{x^{\mu=0,1,2}} \times \underbrace{T^*C} \times \mathbb{R}^3_{7,8,9}$$

T^*C



x^{4+i5} = local holomorphic coord. in fiber of T^*C

x^{6+i10} local holomorphic coord. along C

[we left out x^3. The reason is that this choice of labeling facilitates the comparison with the M-theory description.]

Then

$$\boxed{\phi = X_{4+i5}\, dx^{6+i10}}$$

is a (1,0)-form on C valued in ad P

while $X_{7,8,9}$ remain scalars.

The supersymmetry equations are then really just the ASD instanton equations on T^*C reduced along the fibers - This was essentially Hitchin's original point of view!

In any case, we claim there is a moduli space of $Poin(\mathbb{M}^{1,2})$-invariant vacua in the quantum theory which is precisely Hitchin moduli space $\mathcal{M}_H$.

Now, what is the very low energy dynamics of this theory?

<u>General principle</u>: When there is a moduli space of vacua the LEET is a σ-model whose target space is that moduli space of vacua.

Note that we can take area$(C) \to 0$ to get a 3D field theory.

Our moduli space of vacua is independent of the Kähler class of C and is hence unaffected.

In our case we therefore have a 3D nonlinear sigma model of maps

$$\varphi : M^{1,2} \longrightarrow \mathcal{M}_H$$

It has 8 unbroken supersymmetries. General theorems tell us the target space must have a HK structure, and $\mathcal{M}_H$ indeed carries a HK structure.

In fact, it has a 1-parameter family parametrized by g_s.

(Digression: There is a generalization of this discussion where

$$T^*C \longrightarrow \mathcal{L}_1 \oplus \mathcal{L}_2 \longrightarrow C$$

$$\mathcal{L}_1 \otimes \mathcal{L}_2 \cong K_C$$

$\phi_i \in \Gamma(\mathrm{End}E \otimes \mathcal{L}_i)$, $\mathcal{L}_i$ carry Herm. metrics

w.r.t. canonical connection

$$\bar{\partial}_A \phi_i = 0$$

also $[\phi_1, \phi_2] = 0$

also $F + h_1[\phi_1^\dagger, \phi_1] + h_2[\phi_2^\dagger, \phi_2] = 0$.

This leads to $N=1$ theories in 4D and there is a small community working on these equations.

See:

Yonekura, 1310.7943 section 2.3 and ref's therein for further details.)

5. M-Theory

To go further we need to use some more ideas about string theory.

M-Theory = hypothetical quantum theory in 10 AND 11-dimensional Spacetimes with 32 supercharges in the $\mathbb{M}^{1,10}$ vacuum.

LEET: 11D sugra has fields

$$g_{MN} \in Met(\mathcal{X}_{11}) \quad \psi \in \Gamma(S \otimes T^*\mathcal{X}_{11})$$

$$G \in \Omega^4(\mathcal{X}_{11}) \quad \left(\text{really } \check{H}^4_{\frac{1}{2}W_4}(\mathcal{X}_{11})\right)$$

$$M/\mathcal{X}_{11} = \mathcal{X}_{10} \times S^1_R \xrightarrow[R \to 0]{} IIA/\mathcal{X}_{10}$$

$$\ell_{Pl},\ R \qquad\qquad g_{st},\ \ell_s$$

M-Theory has branes with $p = 2, 5$

(A) M5 with wv. $W^5 \times S^1_R$

$\Longleftrightarrow$ D4 with wv W^5

(B) M2 with wv. $W^2 \times S^1_R$

$\Longleftrightarrow$ Fund. String with wv W^2

Match tensions:

$$(A): \quad \frac{R}{l_{Pl}^6} = \frac{1}{g_s l_s^5} \qquad (B): \quad \frac{R}{l_{Pl}^3} = \frac{1}{l_s^2}$$

$$\Longrightarrow \quad l_{Pl}^3 = g_s l_s^3 , \quad R = g_s l_s$$

Side
<u>Remark</u>: Note that if $g_s \to \infty$ at fixed l_s then $R \to \infty$. This suggests that the 10-dimensional string theory at strong coupling is in fact an 11-dim'l theory.

The LEET on M2, M5 branes is NOT just SYM! It is more subtle.

Again we can distinguish a <u>single</u> brane ("N=1") from <u>multiply-wrapped</u> branes ("N>1").

For N=1 we can write explicit field representations and Lagrangians.

For N=1 M5 on $\mathcal{W}_6 \hookrightarrow \mathcal{X}_{11}$ we can describe the worldvolume theory in terms of a "six-dimensional tensormultiplet":

$$X \in \Gamma(\mathcal{N}) \qquad \mathcal{N} = \text{rank 5 normal bundle}$$

$i : \mathcal{W}_6 \hookrightarrow \mathcal{X}_{11}$

Again these describe motion of the brane in transverse directions.

However, instead of the Maxwell gauge field we have a gerbe connection.

It has a "fieldstrength" or "curvature"

$$H \in \underbrace{\Omega^3_{\mathbb{Z}}(\mathcal{W}_6)}$$

3-forms with integral periods.

Note $\Longrightarrow$ $dH = 0$

So locally $H = dB$ but B is only locally defined. (analog of Maxwell conn. A)

⌈ Really the isomorphism class of the gerbe connection is an element of Deligne-Cheeger-Simons differential Cohomology $\check{H}^3(\mathcal{X}_6)$ ⌋

In addition, in 6-dimensions with Lorentzian metric $* : \Omega^3(\mathcal{W}_6) \rightarrow \Omega^3(\mathcal{W}_6)$ is an involution so we can impose

$$H = *H$$

So our classical equations of motion

$$dH = 0 \quad \& \quad H = *H$$

In the tensormultiplet we also have fermions (analog of gauginos)

$$\psi \in \Gamma[(S(TW) \otimes S(N))^{+}]$$

As an illustration of what is meant by

$$M5/W_5 \times S^1_R \xrightarrow[R \to 0]{} D4/W_5$$

let's just consider the KK reduction of the self-dual 3-form.

When $W_6 = W_5 \times S^1_R$

we can Fourier-decompose our fields in the circle coordinate $\theta \sim \theta + 2\pi$

$$H(x^0, \ldots, x^4, \theta) = \sum_{n \in \mathbb{Z}} H_n(x^0, \ldots x^4)\, e^{in\theta}$$

$n \neq 0$ are massive modes $\sim \dfrac{n^2}{R^2}$

We can decompose H_0 as

$$H_0 = F \wedge d\theta + *_6 (F \wedge d\theta)$$

$$= F \wedge d\theta + (*_5 F) \cdot \frac{1}{R^2} \qquad \left(\begin{array}{c}\text{Assume} \\ \text{Product} \\ \text{metric}\end{array}\right)$$

Then $dH_0 = 0 \iff$

$$dF = 0 \;\; \& \;\; d(*_5 F) = 0$$

These are the Maxwell equations of the U(1) D4 theory.

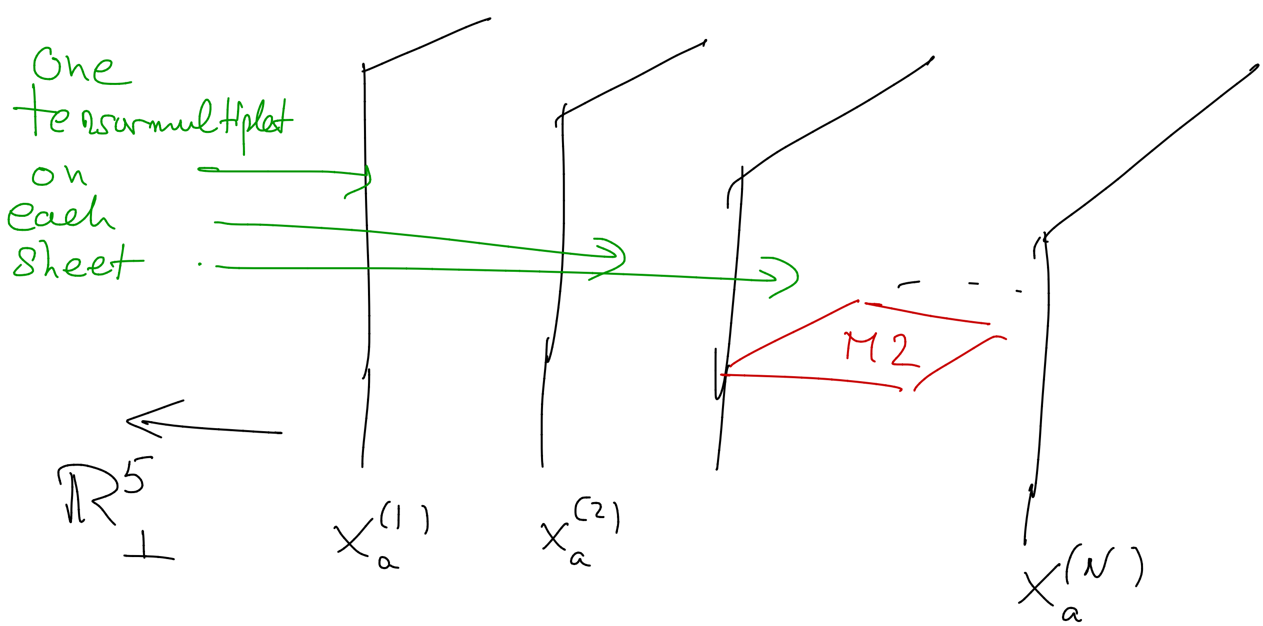
But the situation for N M5 branes is <u>much more subtle</u>.

It is believed that

1.) At energies $E \ll \ell_{Pl}^{-1}$ the LEET is not just an effective theory but a "UV complete" 6D quantum field theory. It is known as the (2,0) $u(N)$ theory. (See below for further remarks).

2.) When $N>1$, just as for D-branes

This theory has a "Coulomb branch", corresponding to N parallel separated M5's



It is called a "Coulomb branch" because, in a sense, the gauge algebra of the tensormultiplet is $U(1)$.

For example, the analog of Wilson lines are

$$\exp\left(2\pi i \int_{X_2} B\right) \qquad X_2 \in Z_2(\mathcal{X}_6)$$

$B \sim$ "gerbe connection" and there are valued in $U(1)$.

(Note that M2 ends on a string in $\mathcal{W}_6$ and these strings couple to gerbe connections.)

3.) In fact, when reduced along a circle in the $R \to 0$ limit

$$\mathcal{W}_6 = \mathcal{W}_5 \times S^1_R$$

The (2,0) Theory reduces to the $U(N)$ SYM on $\mathcal{W}_5$. The M2-branes reduce to fundamental strings and we recover the previous D-brane picture of the Coulomb branch, associated to a stack of D4 branes.

Remark (1) above amplified:

To define a "UV complete theory" – that is, a completely consistent quantum field theory that is well-defined at all distance scales and energies one needs the existence of a conformal field theory.

It is known ("Nahm's theorem") that there is a limited list of superconformal algebras.

If we take a superconformal algebra to be:

a.) A superalgebra $\mathfrak{g}^0 \oplus \mathfrak{g}^1$ such that $\mathfrak{g}^0 \supset so(d+2,2) \oplus \cdots$

b) $\mathfrak{g}^1$ transforms spinorially under $\mathfrak{g}^0$

Then we find that we must use special isomorphisms of Lie algebras defined by Clifford algebras

These only exist in low dimensions and the largest dimension with a superconformal algebra uses so(8) triality (rather so(6,2) triality) and hence is in 6 spacetime dimensions

6. M5 in T^*C : Class $\mathcal{S}$ Theories

We now consider the M-theory geometry

$$\mathcal{X}_{11} = \mathbb{M}^{1,3} \times T^*C \times \mathbb{R}^3_{7,8,9}$$

Again we take the obvious product metric with the HK metric on T^*C.

The latter preserves only 2 out of 4 susy's So of the original 32 susy's (from the real rank of the spin bundle) only 1/2 are preserved and we have 16 unbroken susys.

Now we put N coincident M5-branes on $\mathbb{M}^{1,3} \times C$ where $C \hookrightarrow T^*C$ as the zero section and again $\vec{X}_{7,8,9} = 0$.

The discussion at this point is very similar to that for D4:

The rank 5 normal bundle splits

$$\mathcal{N} \cong \mathcal{N}_2 \oplus \mathcal{N}_3$$

$$\mathcal{N}_2 = \mathcal{N}(C \hookrightarrow T^*C) \qquad \mathcal{N}_3 \cong \mathbb{R}^3_{7,8,9}$$

There is a reduction of structure group

$$\mathrm{Spin}\,5 \longrightarrow \mathrm{Spin}\,2 \times \mathrm{Spin}\,3$$

and the HK metric sets $\nabla^R \cong \nabla^{LC}$

so we have partial top twisting — preserving 8 susy's.

For certain quantities (like the geometry of the space of vacua) the partial topological twisting guarantees independence from the Kähler structure of C.

One might well wonder how we can prove that if we can't even write fundamental field multiplets . . .

The response of a QFT to a change of metric $\delta g_{\mu\nu}$ is to insert

$$\delta g^{\mu\nu} T_{\mu\nu}$$

into correlation functions.

The response to a change in Kähler structure of C is

$$\delta g^{z\bar{z}} T_{z\bar{z}}$$

Now, we <u>do</u> have a stress-energy multiplet with known susy turns and $T_{z\bar{z}}$ will be Q-exact under some unbroken supersymmetries.

Thus, we can consider taking the limit $\text{area}(C) \longrightarrow 0$.

The claim is, aside from some special low-genus cases, like $C = \mathbb{CP}^1$

with no punctures, the limit exists and is a well-defined UV complete <u>FOUR-DIMENSIONAL</u> QFT. This is the definition of class S theories.

<u>If</u> we assume this physical picture makes sense we can then go on to deduce many nontrivial pure math statements. Many such statements have been independently checked.

Two important generalizations:

1.) $U(N)$ gauge group can be generalized to any compact group all of whose roots have equal length:

Actually: only Liealgebra is visible $\mathfrak{g}$

2.) C can have punctures, and at the punctures we must insert boundary conditions, or, better real codimension 2 ½ BPS 4-diml objects in the 6d (2,0) Theory

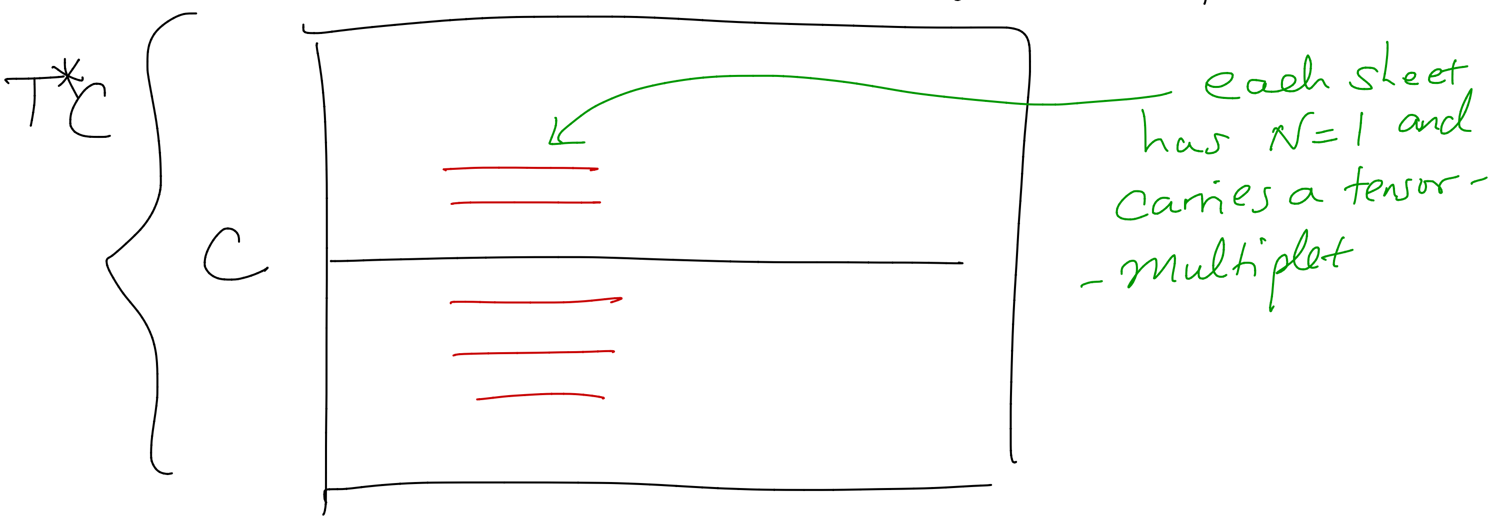
Data of class S theory:

- $\mathfrak{g}$: Lie algebra, all roots $\alpha^2 = 2$
- C: Riemann surface with punctures
- $\mathcal{D}$: data @ punctures

We'll discuss $\mathcal{D}$ a bit more in Lecture 2.

Now we consider the Coulomb branch of vacua of the U(N) class S theory

We split the N M5's in the normal direction and get a picture:



So we get an N-fold cover. But because of supersymmetry we should have a smooth holomorphic curve in T^*C,

So, we get a <u>branched cover</u>

$$\pi: \Sigma \longrightarrow C \,, \quad \Sigma \subset T^*C$$

An important new feature compared to our previous example with 16 susy's is that for a generic point on the Coulomb branch Σ is connected.

The Coulomb branch $\mathcal{B}$ is the space of normalizable deformations of Σ. A deformation is normalizable if

$$\int_{\Sigma} \delta\lambda \wedge \overline{\delta\lambda} < \infty$$

Exercise:

a.) Consider $\lambda^2 = \left(z^3 - 3\Lambda^2 z + u\right) dz^{\otimes 2}$, corresponding to a Hitchin system on $\mathbb{CP}^1$ with one irreg. sing. point at $z=\infty$. Show that Λ is a non-normalizable parameter and u is a normalizable parameter so

$$u \in \mathcal{B} = \mathbb{C}$$

b.) Consider

$$\lambda^2 = \left(\frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z} \right) dz^{\otimes 2}$$

Corresponding to an ISP at $z=0,\infty$ on $\mathbb{CP}^1$. Again Λ is non-normalizable and u is normalizable and $\mathcal{B} = \mathbb{C}$

c.) Consider the case $C = E - \{0\}$ with RSP

$$\lambda^2 = \left(m^2 \wp(z|\tau) + u\right)(dz)^2$$

τ, m are non-normalizable

u is normalizable

7. Coulomb Branch of $d=4, N=2$ Theory

Should start with basic data

Now we discuss some general facts about $d=4, N=2$

The LEET of the Coulomb branch of a $d=4, N=2$ theory includes a self-dual Abelian Maxwell theory:

- $\mathbb{V}$ Real symplectic vector space $\dim_{\mathbb{R}} = 2r$ with compatible complex structure $J(u)$

$u \in \mathcal{B}$ = Coulomb branch. ($\Rightarrow$ special Kähler)

Symplectic + complex structure are compatible $\Rightarrow$ we get a metric on $\mathbb{V}$, and we require it to be > 0.

- $\mathbb{F} \in \Omega^2(\mathbb{M}^{1,3}; \mathbb{V})$

$$d\mathbb{F} = 0 \quad \& \quad \mathbb{F} = *_{\mathbb{M}^{1,3}} \otimes J(u)\, \mathbb{F}$$

- A choice of maximal Lagrangian decomposition $\mathbb{V} = L \oplus L^{\perp} \Longrightarrow$ Lagrangian description

$$\langle \alpha_I, \alpha_J \rangle = \langle \beta^I, \beta^J \rangle = 0 \quad \langle \alpha_I, \beta^J \rangle = \delta_I^{\ J}$$

$I, J = 1, \dots, r$

$$\mathbb{F} = \alpha_I F^I + *(\alpha_I F^I)$$

$$dF^I = 0$$

Symp. str. + Complex str. + duality basis $\Rightarrow$

τ_{IJ} : in Siegel upper half-plane

$$S = \int \operatorname{Im}\tau_{IJ} F^I * F^J + \operatorname{Re}\tau_{IJ} F^I F^J$$

Magnetic monopoles & dyons $\Rightarrow V$ has an underlying integral structure

$$V = \Gamma \otimes \mathbb{R}$$

Γ = lattice with integral antisymmetric pairing (Dirac pairing)

Now, when describing the LEET on the Coulomb branch of $d=4$ $N=2$ theory all of this is parametrized by $u \in \mathcal{B}$

$$\left.\begin{array}{c}\Gamma \\ \downarrow \\ \mathcal{B}^* \subset \mathcal{B}\end{array}\right\}$$ local system

$\mathcal{B} - \mathcal{B}^*$ = divisor around which there is monodromy

In addition, $N=2$ susy $\Rightarrow$ there is an extra piece of data: The $N=2$ central charge

$$Z : \Gamma \longrightarrow \mathbb{C}$$

Z is linear on the fibers

When we choose (locally) a duality frame $\{\alpha^I, \beta_I\}$ for Γ we can define

Special coordinates: $I = 1, \dots, r$

$$a^I = Z(\alpha_I) \qquad a_{D,I} = Z(\beta^I)$$

$N=2 \Longrightarrow$ Lagrangian condition:

Locally $\exists\, \mathcal{F}$ $\quad a_{D,I} = \dfrac{\partial \mathcal{F}}{\partial a^I}$

$$\tau_{IJ} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J}$$

So, we get a family of PPAV over $\mathcal{B}$ of same dimension as $\mathcal{B}$

Bosonic LEET:

Now promote a^I to <u>fields</u> on $M^{1,3}$

Conceptually: These describe a σ-model of maps $M^{1,3} \longrightarrow \mathcal{B}$ and we can write a Lagrangian for maps into a coordinate patch with special coordinates. The LEET is:

$$S = \frac{1}{4\pi} \int \Big\{ \mathrm{Im}\,\tau_{IJ} \left(da^I * da^J + F^I * F^J \right) - \mathrm{Re}\,\tau_{IJ}\, F^I \wedge F^J + \text{Fermions} \Big\}$$

(Note $\mathrm{Im}\,\tau_{IJ} > 0$ required for sensible action...)

Define Seiberg_Witten moduli space

$$\mathcal{M} \longrightarrow \mathcal{B}$$

$$\mathcal{M}_u = \Gamma_u^\vee \otimes \left(\mathbb{R}/2\pi\mathbb{Z} \right)$$

To "solve for the vacuum structure" of an $N=2$, $d=4$ theory is to give this data of a holo family of PPAV with central charge function.

Seiberg-Witten and much subsequent work found that – in all known cases – there is a holomorphic family of Riemann surfaces

$$\begin{array}{ccc} \Sigma_u & & \Sigma \\ \downarrow & & \downarrow \\ u & \hookrightarrow & \mathcal{B} \end{array}$$

equipped with meromorphic diffl λ_u on Σ_u (varying holomorphically with u)

- Γ_u = Subquotient of $H_1(\Sigma_u, \mathbb{Z})$
- $Z(\gamma) = \oint_\gamma \lambda_u \qquad \gamma \in \Gamma_u$
- M_u = Jacobian or Prym

We stress that at this point

- There is no a-priori reason why the data should come from a holo family (Σ_u, λ_u) $u \in \mathcal{B}$

- There is no assertion that $\mathcal{M}$ is a Hitchin moduli space. In fact, it is quite likely that there are many $d=4$, $\mathcal{N}=2$ theories where $\mathcal{M}$ <u>cannot</u> be described by Hitchin moduli space.

- All that the constraints of $d=4, \mathcal{N}=2$ give us is a local system over a S.K. mfld

$$\begin{array}{ccc} \Gamma_u & \hookrightarrow & \Gamma \\ \downarrow & & \downarrow \\ u & \hookrightarrow & \mathcal{B}^* \end{array}$$

Γ has antisymmetric $\mathbb{Z}$-pairing

and $Z : \Gamma \longrightarrow \mathbb{C}$ $\mathcal{N}=2$ central charge function $\Longrightarrow$ $\Gamma^\vee \otimes \mathbb{R}/2\pi\mathbb{Z}$ is a family of PPAV

8. Recovering Seiberg-Witten Theory

Now we consider the Abelian tensor-multiplet theory on $M^{1,3} \times \Sigma$.

If we take area$(\Sigma) \longrightarrow 0$ then we first note that KK reduction of the self-dual gerbe connection gives us the structure of a self-dual Abelian gauge theory:

We have: $dH = 0$ and $H = * H$

We set $V = \mathcal{H}^1(\Sigma) =$ harmonic 1-forms

(underlying integral structure $H^1(\Sigma, \mathbb{Z}) \cong H_1(\Sigma, \mathbb{Z})$)

Choose a symplectic basis $\{\alpha_I, \beta^I\}$ for V and use KK:

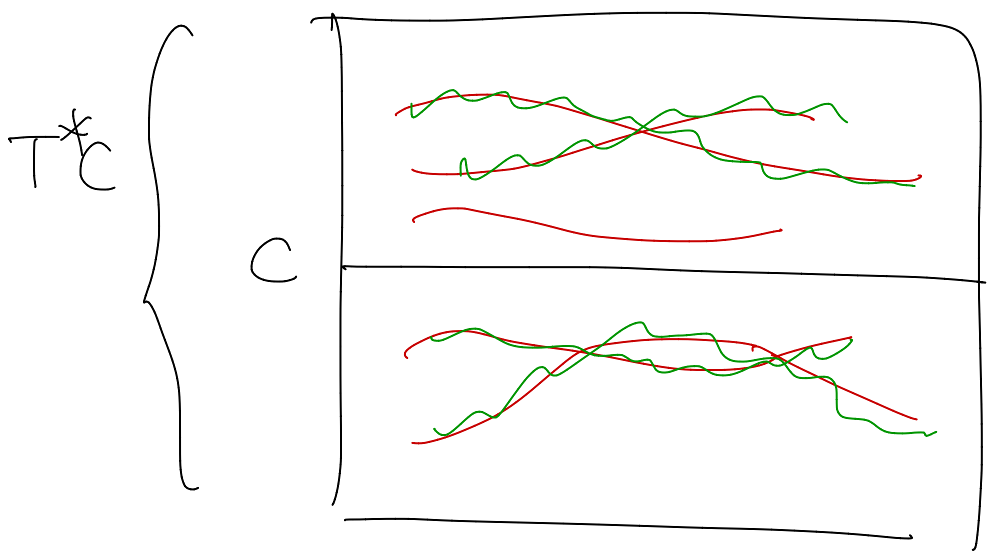
$$H = \underbrace{\alpha_I F^I + *_{M^{1,3}} \otimes J(u)(\alpha_I F^I)}_{\text{same data as } \mathbb{F}.}$$

So the LEET in $M^{1,3}$ is that of a self-dual abelian gauge field for V with complex structure inherited from Σ.

Similarly, there is an action for the single M5 and it involves the induced volume

$$\text{Action} = \frac{1}{\ell_p^6} \int_{W^6} d^6\xi \sqrt{\det\left(\iota^* g_{MN} + \cdots\right)} + \cdots$$

Fluctuations around a holomorphic curve



are described by (S_0 = original volume, it might be ∞)

$$S - S_0 = \int_{M^{13}} \mathrm{Im}\,\tau_{IJ}(a)\, da^I \wedge * da^J + \cdots$$

Details of this computation can be found in Gaiotto-Moore-Neitzke, 0907.3987 sec 3.1.5

It is nice to check our identification of Σ_u with the SW-curve and λ_u with the SW-differential by looking at "BPS states."

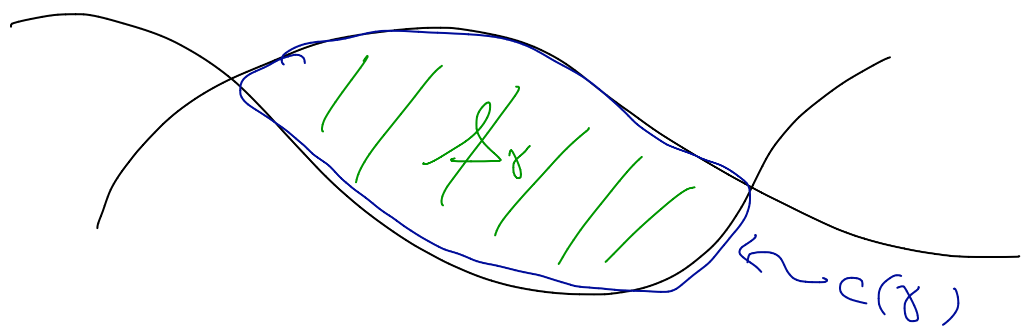
There are many ways to think of BPS states.

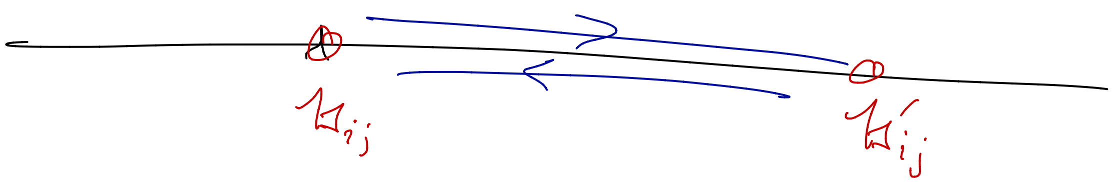
The charge lattice is $\cong H_1(\Sigma_u, \mathbb{Z}) = \Gamma_u$

For a given charge $\gamma \in H_1(\Sigma_u, \mathbb{Z})$ we consider the length-minimizing curve $c(\gamma) \subset \Sigma_u$ and the area-minimizing surface $S_\gamma \subset T^*C$ so that

$$\partial S_\gamma = c(\gamma)$$

For example, if we have 2 (ij) branch points we might have





In M-theory the quantization of the moduli space of such S_γ gives the space of BPS states of charge γ.

The central charge is

$$Z = \int_{S_\gamma} \Omega^{2,0} = \int_{S_\gamma} d\lambda = \oint_\gamma \lambda$$

So we recover the SW formula for the $N=2$ central charge

We conclude that

- $\mathcal{B}$ = space of normalizable deformations of Σ
- $\Sigma_u \subset T^*C$ is the SW curve
- Restriction λ_u of Liouville form is the SW differential.

Note: We can make contact with the introductory remarks on spectral networks

The length minimizing condition says

$$\left| \int \lambda_{ij} \right| = \int |\lambda_{ij}|$$

so $\langle \partial_t, \lambda_{ij} \rangle$ has constant phase

but then

WLOG we can rescale t so that

$$\langle \partial_t, \lambda_{ij} \rangle = e^{i\vartheta_*}$$

Now we see that we have

* Coinciding ij and ji curve connecting branch points
* $Z_\gamma = \oint_\gamma \lambda = e^{i\vartheta_*} |Z_\gamma|$

So the phase where finite WKB networks occur are, indeed the same as the phases at which the SN's jump discontinuously

<u>Side Remark</u>: There are many different ways of representing the space of BPS states. One particularly nice way, which is very clearly defined mathematically, makes use of L^2 kernels of Dirac operators on moduli spaces of magnetic monopoles. See end of Lecture 2.

9. Relation To Hitchin Moduli Space

Now we wish to connect the D4 and M5 stories using the principle

$$M5/\mathbb{W}_5 \times S^1_R \longrightarrow D4/\mathbb{W}_5$$

So we modify the $\mathbb{M}^{1,3}$ in $\mathcal{X}_{11}$ above to $\mathbb{M}^{1,2} \times S^1_R$.

$$\mathbb{M}^{1,3} \longrightarrow \mathbb{M}^{1,2} \times S^1_R$$

$$\mathcal{X}_{11} = \mathcal{X}_{10} \times S^1_R$$

$$\mathcal{X}_{10} = \mathbb{M}^{1,2} \times T^*C \times \mathbb{R}^3_{7,8,9}$$

exactly as in our discussion of D4 branes.

Now from

$$\mathbb{M}^{1,2} \times S^1_R \times T^*C \times \mathbb{R}^3_{7,8,9}$$

with area $(C) \longrightarrow 0$ we get the LEET of the class S theory compactified on $\mathbb{M}^{1,2} \times S^1_R$.

But the compactification of the theory

$$\int_{\mathbb{M}^{1,2} \times S^1_R} \mathrm{Im}\,\tau_{IJ} \left(F^I * F^J + da^I * d\bar{a}^J \right) + \mathrm{Re}\,\tau_{IJ} F^I_a F^J$$

gives a 3D HK σ-model with <u>semiflat</u> <u>metric</u>:

$$\longrightarrow \int_{\mathbb{M}^{1,2}} R\,\mathrm{Im}\,\tau_{IJ}\, da^I d\bar{a}^J + R^{-1} (\mathrm{Im}\,\tau)^{-1,IJ}\, dz_I\, d\bar{z}_J$$

$$dz_I = d\varphi_{m,I} - \tau_{IJ}\, d\varphi_e^J$$

Details of Computation

$M^{1,2} \times S^1$ metric $ds^2 = dx^\mu dx_\mu + R^2 (dx^3)^2$

$x^\mu = x^{0,1,2}$ $\quad x^3$

$$x^3 \sim x^3 + 2\pi$$

KK reduce:

$$a^I \longrightarrow a^I(x^\mu)$$

$$A^I \longrightarrow \varphi_e^I(x^\mu)\, dx^3 + \bar{A}^I$$

$$F^I = d\varphi_e^I\, dx^3 + \bar{F}^I$$

$$\varphi_e^I(x^\mu) = \oint_{S^1} A^I \sim \varphi_e^I(x^\mu) + 1$$

$$da^I *_4 d\bar{a}^J = R\, dx^3\, da^I *_3 d\bar{a}^J$$

$$F^I *_4 F^J = \frac{dx^3}{R} d\varphi_e^I *_3 d\varphi_e^J + R\, dx^3\, \bar{F}^I *_3 \bar{F}^J$$

Integrate x^3 to get

$$\int_{M^{1,2}} -\frac{R}{2} \operatorname{Im}\tau_{IJ}\, da^I *_3 d\bar{a}^J - \frac{1}{2R} \operatorname{Im}\tau_{IJ}\, d\varphi_e^I *_3 d\varphi_e^J$$

$$- \frac{R}{2} \operatorname{Im}\tau_{IJ}\, \bar{F}^I *_3 \bar{F}^J - \operatorname{Re}\tau_{IJ}\, \bar{F}^I d\varphi_e^J$$

Now dualize the 3D gauge field $\bar{F}^I$ to a periodic scalar. This is a kind of Fourier transform:

$$\int d\bar{F}^I d\varphi_{m\,I} \exp\left\{ i\int -\frac{R}{2}\,\mathrm{Im}\,\tau_{IJ}\, \bar{F}^I *_3 \bar{F}^J \right.$$

$$\left. + i\int \bar{F}^I(d\varphi_{m\,I} - \mathrm{Re}\,\tau_{IJ}\, d\varphi_e^J) \right\}$$

Integrating out $\varphi_{m,I}$ recovers previous Theory: It says $\bar{F}^I$ is a closed 2-form with integer periods.

On the other hand we could instead integrate out $\bar{F}^I$ using a Gaussian integral.

<u>Stationary point:</u>

$$\bar{F}^I = -\frac{1}{R}(\mathrm{Im}\,\tau)^{-1,IJ}\left(d\varphi_{m\,J} - \mathrm{Re}\,\tau_{JK}\, d\varphi_e^K\right)$$

plug in and get above action ∎

So: Further compactification of the class S theory to 3D gives a σ-model with 8 supersymmetries with HK target:

$$\begin{array}{ccc} \mathcal{J}_u & \hookrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \pi \\ u & \hookrightarrow & \mathcal{B} \end{array}$$

On the other hand, from the D4-brane perspective we now know that $\mathcal{M}$ should be $\mathcal{M}_H$.

Summary

$u(N)$ $(2,0)$ on
$\mathbb{M}^{1,2} \times S^1_R \times C$

$\text{area}(C) \ll R^2$ $\swarrow$ $\qquad$ $\searrow$ $R^2 \ll \text{area}(C)$

$d=4$ class S theory $S[\mathfrak{g}, C, D]$
$\downarrow R \to 0$
HK σ-model $\mathbb{M}^{1,2} \to \mathcal{M}_{SW}$

U(N) Sym on C
Moduli of vacua $= \mathcal{M}_H$
$\downarrow \text{area}(C) \to 0$
HK σ-model $\mathbb{M}^{1,2} \to \mathcal{M}_H$

HK σ-model $\mathbb{M}^{1,2} \to \mathcal{M}_{SW}$ $=$ HK σ-model $\mathbb{M}^{1,2} \to \mathcal{M}_H$

Of course the s.f. metric will get quantum corrections from BPS particles.

This vacuum geometry should work at all R and for this quantity, effectively

$$M5 / W_5 \times S^1_R = D4 / W_5$$

This is why the exact HK metric on Hitchin moduli space is exponentially close to the s.f. metric (for large R) and can, in principle, be computed exactly following the procedure in

Gaiotto - Moore - Neitzke, 0807.4723.